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A FAMILY OF NEIGHBORLY POLYTOPES

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ABSTRACT

A d-polytope P is said to be neighborly provided each $[d/2]$ vertices determine a face of P. We construct a family of d-polytopes that are dual to neighborly polytopes by means of facet splitting. We use this family to find a lower bound on the number of combinatorial types of neighborly polytopes. We also show that all members of this family satisfy the famous Hirsch conjecture.

1. Introduction

A d-polytope is said to be *neighborly* provided each set of $[d/2]$ vertices determines a face of P. One family of neighborly d-polytopes is the family of cyclic d-polytopes. A *cyclic* d-polytope is defined to be the convex hull of $d+1$ or more points on the curve in R^d consisting of all points of the form $(t, t^2, \cdots, t^d)$, where t runs through the real numbers. This curve is called the *moment curve.* Remarkably, the combinatorial type of the cyclic d-polytope depends only on the number of points chosen, and not on their location on the curve. This gives us an infinite family of neighborly d-polytopes with one combinatorial type for each possible number of vertices.

It is known that there are other neighborly d-polytopes (i.e., ones that are not isomorphic to any cyclic polytope), but there has not appeared in the literature any systematic construction of other infinite families of neighborly d-polytopes. The closest to this is in Grünbaum's book, *Convex Polytopes* [3], where he constructs a non-cyclic neighborly 4-polytope with eight vertices, and then remarks that the same methods can clearly be used to get examples with larger numbers of vertices and also examples in higher dimensions.

In this paper we construct an infinite family of duals of neighborly d-polytopes by using facet splitting. This family is much richer in combinatorial types than the duals of the cyclic polytopes. The number of different combinatorial types

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with n facets grows arbitrarily large as n increases. We will get an estimate on the number of combinatorial types in our family, thus establishing a lower bound on the number of combinatorial types of neighborly d-polytopes with n vertices. We shall also show that the polytopes in our family have diameter at most $n - d$, where n is the number of facets, thus confirming the Hirsch conjecture for these polytopes.

2. Definitions and preliminaries

If a polytope is the dual of a neighborly d-polytope, we shall call it a **-neighborly polytope.* Clearly, a *-neighborly d-polytope is one in which each $[d/2]$ facets have nonempty intersecton. When the polytope is simple, that is, it is the dual of a polytope whose faces are all simplices, then we can determine the dimension of this nonempty intersection.

LEMMA 1. *In a simple polytope, if k facets have a nonempty intersection, then the dimension of their intersection is $d - k$.*

Another fact about simple d-polytopes that we shall need is the following:

LEMMA 2. *If, in a simple d-polytope P, a k-face H meets a facet F, and if $H \not\subset F$, then $G = H \cap F$ is a face of dimension $k - 1$.*

Lemmas 1 and 2 are standard results. The reader should consult [3, chapter 3].

Our polytopes will be constructed by *facet splitting*. For simple 3-dimensional polytopes, facet splitting is accomplished by doing the following. We choose a segment across a facet F of a polytope P, missing the vertices of the facet, thus cutting the facet into two polygons. Next, a plane H containing this segment is chosen such that H separates the vertices of one of the polygons, other than the vertices on H, from the other vertices of the polytope (see Fig. 1).

Let H^+ be the closed halfspace with boundary H, containing the vertices of P that are not on the facet being split. Let P' be the polytope $P \cap H^+$. We say that P' is obtained from P by splitting F.

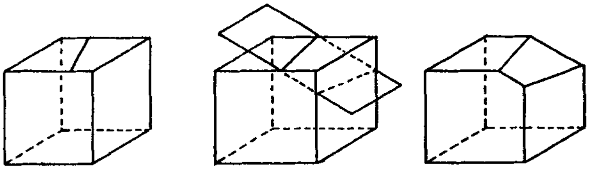

Fig. 1.

Whenever this splitting is performed on simple polytopes the two new facets will be isomorphic to the polygons whose union is F. Furthermore, the two new facets will meet the other faces of P' the same way that the two polygons meet the faces of P. In other words, we can see the combinatorial structure of the new polytope without actually using the plane H to cut off part of P.

For a simple d-polytope P, the same process is used. A facet is chosen to be split. A $(d-2)$-dimensional hyperplane H in the $(d-1)$-dimensional affine hull of that facet is chosen that misses the vertices of P and separates the facet into two $(d-1)$-polytopes. A hyperplane (in R^d) is chosen that contains H, and separates the vertices of one of the two $(d-1)$-polytopes, other than vertices on H, from vertices of P. Let H^+ be the closed halfspace bounded by H, containing the vertices of P that are not on F. The intersection of P with H^+ is the polytope that we say is obtained from P by *splitting the facet* F. A careful accounting of facet splitting and its properties can be found in [2].

Splitting a facet F of a polytope P, induces facet splittings on facets of P that meet F. Let F' be a facet of P that meets F on a subfacet S that is intersected by H. The intersection of H with the affine hull of S is a hyperplane in the affine hull of S, separating S into two $(d-2)$-polytopes. The intersection of the affine hull of S with the hyperplane that "cuts off" part of F from the rest of P, also will "cut off" the vertices of one of the two $(d-2)$-polytopes from the rest of F'. This is what we call a splitting *induced* by the splitting of F. Note that this induced splitting is a splitting of dimension one lower than the original splitting. This induced splitting will induce splittings of still lower dimensional faces in the same way.

It is important that the original polytope is simple, for otherwise the two new facets will not be isomorphic to the two $(d-1)$-polytopes into which F was separated. This splitting process produces a simple polytope, thus one can perform repeated facet splittings on any given simple polytope.

There are several special classes of 3-polytopes that we shall need to know about. The first is the class of *based* 3-polytopes. These are the simple 3-polytopes that have one facet, called the base, that meets all other facets. When the graph of such a polytope is drawn in the plane with the base as the outside face, it looks like a simple circuit for the outside face with the vertices and edges inside the circuit forming a tree. Because of this, it is easy to show that there is always a triangular face meeting the base. Using the existence of these triangular faces, one can easily prove the following:

LEMMA 3. *The based 3-polytopes can be generated by repeated facet splittings* *starting with the tetrahedron. These facet splittings being of the special type where one of the two polygons that the facet is separated into, is a triangle meeting the base.*

This special type of splitting is usually referred to as *truncating a vertex*, because the part of the polytope that is cut off is just a vertex.

The second class of 3-polytopes we shall call *double based* polytopes, for they are the simple 3-polytopes that have two facets that meet all of the other facets. Figure 2 shows the first four members of this class.

One property of cyclic polytopes that we shall need is *Gale's "evenness condition"*. This is a condition that determines which vertices of a cyclic d-polytope are the vertices of the facets. The condition is simply that any set of d vertices of a cyclic d-polytope P determines a facet of P provided that given any two vertices not in this set of d vertices, there is an even number of vertices of the set between them. Betweenness here is determined by the order of the vertices on the moment curve.

LEMMA 4. *If the vertices of a d-polytope can be ordered in such a way that a set of d vertices is the set of vertices of a facet if and only if it satisfies the evenness condition, then the polytope is isomorphic to the cyclic d-polytope with the same number of vertices.*

PROOF. We simply take the vertices in the given ordering and map them onto the corresponding vertices in the ordering on the moment curve. This mapping is clearly an isomorphism.

Actually, we shall work with an ordering of the facets of the dual polytopes, thus we shall have an ordering of the facets such that a set of d facets determines a vertex if and only if any two other facets are separated by an even number of the d facets. From Lemma 4, it follows that any two simple d-polytopes with the same number of facets that satisfy this evenness condition will be isomorphic.

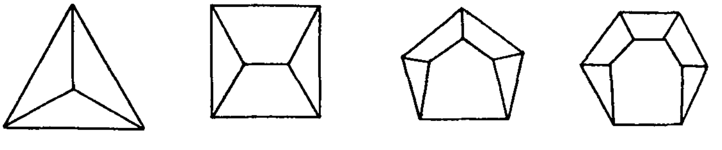

Fig. 2.

3. The 4-s-polytopes

We begin by describing how we generate our *-neighborly 4-dimensional polytopes. The polytopes that we generate will be called d-s-polytopes. The "d" is the dimension of the polytope, the "s" is for "splitting".

The 4-s-polytopes are generated by a sequence of facet splittings starting with the simplex.

We choose a facet F_0 of the 4-simplex and a facet S_0 of F_0. The facet F_0 is split in the following manner: we take a plane in the affine hull of F_0, parallel to S_0, then rotate the plane so that one vertex v_0 of S_0 is on the other side of the plane after the rotation (Fig. 3). This is the plane that we use to define a splitting of F_0. The facet F_0 is split into two facets. The facet that does not contain v_0, is called F_1, and the facet of F_1 that resulted from the induced splitting of S_0, is called S_1.

Inductively, we split F_k, by taking a plane in its affine hull, parallel to S_k, rotating the plane so that one vertex v_k of S_k is on the other side of the plane after the rotation, and using that plane to define the splitting of F_k. The new facet that does not contain the vertex v_k is called F_{k+1}. The new subfacet resulting from the induced splitting of S_k, that does not contain v_k, is called S_{k+1}.

In each of these polytopes, the facet that meets F_k on S_k will be called B_k.

The first two splittings will each produce one combinatorial type of polytope. The third splitting will produce all three types of *-neighborly 4-polytopes (this can be verified by performing the splittings and comparing the results with the known enumerations of these polytopes (see [4])).

In general, many combinatorial types of 4-polytopes will be generated with each splitting. In Theorem 1 we get a lower bound on the number of combinatorial types produced.

LEMMA 5. *The facet F_k is a double based 3-polytope.*

PROOF. This is clearly true for F_0. Proceeding by induction, we assume F_k is a double based polytope. The plane parallel to S_k will split F_k into two 3-

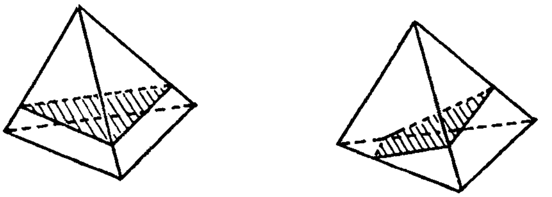

Fig. 3.

polytopes, the one containing v_k being a prism over S_k. Clearly, rotating the plane produces a double based polytope. (Fig. 4.)

LEMMA 6. *Every combinatorial type of based 3-polytope can be realized by one of the facets B_k.*

PROOF. When we split the facet F_k, the induced splitting on B_k is a vertex truncation. Any vertex of S_k can be so truncated, simply by the choice of vertex that the rotating plane passes. By Lemma 3, this generates all of the combinatorial types of based 3-polytopes.

THEOREM 1. *The number of combinatorial types of neighborly 4-polytopes with n vertices is at least*

$$\frac{H(n)}{n}+\frac{n+1}{4n}H\left(\frac{n+1}{2}\right)+\frac{2n+4}{9n}H\left(\frac{n+2}{3}\right)$$

where

$$H(n)=\frac{(2n-6)!}{(n-1)!\,(n-3)!},\qquad \textit{for } n \textit{ an integer,}$$

$$=0 \qquad \textit{otherwise.}$$

PROOF. Let $R(n)$ be the number of combinatorial types of based 3-polytopes with n 2-faces. Rademacher has shown [9] that

$$R(n)=G(n)+\tfrac{1}{2}(n/2+1)G(n/2+1)+\tfrac{2}{3}(n/3+1)G(n/3+1)$$

where $G(n)=(2n-4)!/n!\,(n-2)!$ for n an integer, $=0$ otherwise.

We shall begin by counting rooted 4-s-polytopes. By a *rooted* 4-polytope we mean a 4-polytope together with one of its facets, which we call the *root* of the polytope. Two rooted 4-polytopes will be called *root-isomorphic* provided there is an isomorphism between them that takes one root onto the other. Clearly, two polytopes with non-isomorphic roots are not root-isomorphic. It follows from Lemma 6 that there are at least $R(n-1)$ combinatorial types of roots for *-neighborly 4-polytopes with n facets (note that each facet of such a 4-polytope will have $n-1$ 2-faces).

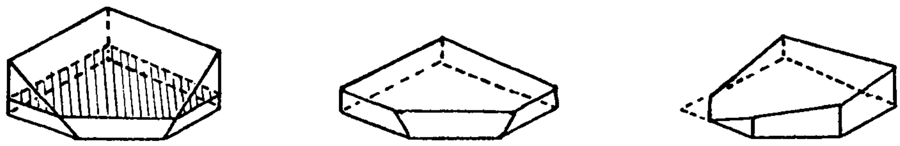

Fig. 4.

To each root isomorphism type there can correspond just one combinatorial type. On the other hand, there can be at most n different root isomorphism types for each combinatorial type — one root type for each choice of a root. We thus have at least $R(n-1)/n$ combinatorial types of 4-s-polytopes, and therefore at least that many *-neighborly 4-polytopes with n facets. Since $R(n-1)/n$ equals the bound in our theorem we are done.

The family of 4-s-polytopes is not the same as the family of *-neighborly 4-polytopes. In their enumeration of neighborly 4-polytopes with 9 vertices, Altshuler and Steinberg [1] found five polytopes with the property that no vertex has a link isomorphic to the dual of any double based polytope. It follows that the duals of these polytopes will not have any facets that are double based polytopes. Since each 4-s-polytope has at least one such facet, namely F_k, we see that there are at least five *-neighborly 4-polytopes with 9 facets that are not 4-s-polytopes.

We can use a systematic notation for the 4-s-polytopes that we generate. For any double based polytope F_k, let us number the vertices of a base face in a cyclic ordering according to a right hand rule (i.e., if the fingers of the right hand point in the direction of increasing numbers, the thumb points upward (Fig. 5)). The sequence of splittings performed can then be denoted by a sequence of integers corresponding to the vertices of S_k that we rotate the plane past. Unfortunately, different sequences of integers can correspond to the same combinatorial type. For example, by symmetry, the sequence $a_1, \cdots, a_k$ is isomorphic to the sequence $n-2-a_1, \cdots, n-2-a_k$ for any 4-s-polytope with n facets. There are other less obvious examples. It is not at all clear to the author how to tell if two sequences correspond to the same combinatorial type.

THEOREM 2. *The duals of the cyclic* 4-*polytopes are* 4-s-*polytopes.*

PROOF. We shall show that the polytopes with the sequences $1, \cdots, 1$ are dual to the cyclic 4-polytopes. For any facet F_k, we number its 2-faces (as illustrated in Fig. 6 for two of the F_k's).

Using this numbering, we induce a numbering of the facets of the 4-polytope.

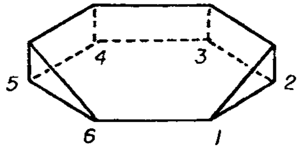

Fig. 5.

Fig. 6.

The facet that meets F_k on a 2-face with the number i will also be given the number i. We complete the numbering of the facets by giving F_k the number $k+5$.

After we have performed a splitting on F_k, we let F_{k+1} have the number $k+6$; we let the other 3-polytope produced from F_k have the number that was assigned to F_k, namely $k+5$, and let all other facets retain the numbers that they had before the splitting. Notice that this induces the same kind of numbering on the facets of F_{k+1} as described above. This is illustrated in Fig. 7 for the case $k=2$.

Now we show that this numbering will give us a Gale's evenness condition for determining vertices of the polytope. Any vertex of F_k that meets a quadrilateral face of F_k is determined by two consecutively numbered facets together with either facets $k+4$ and $k+5$, or facets 1 and $k+5$. In either case, facets not meeting the vertex are separated by an even number of facets meeting the vertex.

The two vertices of F_k that do not meet any quadrilateral faces of F_k are determined by the facets 1, 2, $k+4$ and $k+5$, or else by facets 1, $k+3$, $k+4$ and $k+5$. In each case our evenness condition holds. It remains for us to show that the evenness condition holds for vertices not on F_k. But observe that the condition will hold for all vertices of the 4-simplex, and that all other vertices are on some F_k at some stage of the splitting sequence. Since the numbers assigned to facets do not change, the vertices not on F_k satisfy the evenness condition because they satisfied it earlier in the generation process.

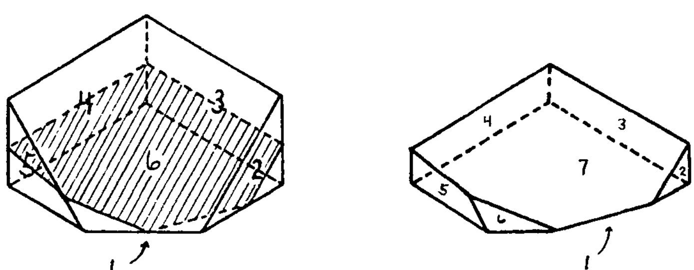

Fig. 7.

In this argument, one might worry that vertices determined by a set of facets which includes the facet numbered $k+5$, might fail to satisfy the evenness condition later in the splitting process. It would seem that this could happen since one can have an odd number of facets at the end of the ordering and still satisfy the evenness condition, but if later on this odd number of facets is not at the end of the ordering, then the evenness condition no longer holds. But fortune is with us, these vertices are on F_k at every stage of the splitting process, and we have given an argument for vertices on F_k that does not depend on the evenness condition ever having held at a previous stage.

Since the evenness condition determines the combinatorial structure, and since the duals of the cyclic 4-polytopes satisfy the evenness condition, we are done.

4. The d-s-polytopes

The description of the generation of the *-neighborly polytopes in d-dimensions is more complicated. The facet splitting is done as follows. Let F^{d-1} be the facet to be split (we shall use superscripts to indicate dimension). We choose a nested set of faces $F^2 \subset F^3 \subset \cdots \subset F^{d-1}$, such that for each k, $3 \leqq k \leqq d-1$, F^{k-1} meets every $[(k+2)/2]$ face of F^k. Of course such a sequence may not be possible for an arbitrarily chosen facet, but we shall show that such a choice is possible in the generation of our family of *-neigborly d-polytopes.

In the affine hull of F^{d-1}, we choose a hyperplane π parallel to F^{d-2} and close enough to it that π separates the vertices of F^{d-2} from the other vertices of F^{d-1}. Any point on the same side of π as F^{d-2} will be said to be *below* π, while points on the other side will be said to be *above* π.

We rotate π until F^{d-3} is above π but the other vertices of F^{d-2} are below. Then we rotate π until F^{d-4} is below π but the other vertices of F^{d-3} are still above π. We continue in this manner until we have rotated π past the face F^2. Whether F^2 is above or below π at this stage depends on the parity of the dimension. If F^2 is above π we choose a vertex v of F^2 and rotate π until v is below π. If F^2 is below π we rotate π until v is above π.

After this sequence of rotations has been performed we have the hyperplane that determines the splitting of F^{d-1}. It might seem that we cannot always perform the required rotations. But observe that whenever we have a $(k-1)$-hyperplane in the affine hull of an F^k separating the vertices of a facet F^{k-1} from the other vertices of F^k, and sufficiently close to F^{k-1}, then that hyperplane can be rotated so that any chosen facet F^{k-2} of F^{k-1} will pass to the other side of the hyperplane. In our sequence of rotations, the intersection of the hyperplane π with the affine hull of F^k will be just such a $(k-1)$-dimensional hyperplane. Thus it is possible to rotate π such that the intersection of π with F^k rotates past F^{k-2}.

LEMMA 7. *π cuts every $[(k+2)/2]$-face of F^k for each k, $3 \leqq k \leqq d-1$.*

PROOF. For $k=3$, this is just the statement that π cuts every 2-face of F^3. Since $\pi \cap F^3$ is a 2-dimensional plane, since the two-dimensional plane rotates past a vertex of F^2, and since F^2 meets each 2-face of F^3, we see that this is just the case we have considered when we were generating the *-neighborly 4-polytopes. The lemma is thus true for $k=3$. We now proceed by induction. Consider an arbitrary face F^k. Let us assume that π has been rotated until it separates the vertices of F^{k-1} from the other vertices of F^k. We use the term *upper face* for the $[(k+2)/2]$-faces of F^k that are not subsets of F^{k-1}, and the term *lower face* for the other $[(k+2)/2]$-faces of F^k.

At this stage in the sequence of rotations, π cuts every upper face of F^k (recall that F^{k-1} meets every upper face of F^k). After the rest of the rotations are performed, it is conceivable that π does not cut all of the upper faces anymore. We shall see that this is not so. Let us assume, without loss of generality, that F^{k-1} lies below π before we do the rest of the rotations. For π to fail to cut an upper face after the rest of the rotations, π would have to lie below the intersection of that upper face with F^{k-1}. By Lemma 2, this intersection is a $([(k+2)/2]-1)$-face of F^{k-1}. Since π is rotated past F^{k-2} but never past the other vertices of F^{k-1}, such a $([(k+2)/2]-1)$-face would have to be a face of F^{k-2}. By induction, however, π cuts every $([(k+2)/2]-1)$-face of F^{k-2}. Thus after all of the rotations π still cuts every upper face.

Now we consider the $[(k+2)/2]$-faces of F^{k-1}. These are the lower faces. By Lemma 2 a lower face X would meet F^{k-2} on a $([(k+2)/2]-1)$-face or else would be a face of F^{k-2}. If X is a face of F^{k-2} then π cuts X because it even cuts every facet of X (by induction). If X intersects F^{k-2} on a $([(k+2)/2]-1)$-face then since we have seen that this face is cut by π we see that X is also cut by π.

For each face F^k, there will be two k-faces created when the facet is split. We define $F^{k'}$ to be the one that does not contain the vertex v.

LEMMA 8. *$F^{2'} \subset F^{3'} \subset \cdots \subset F^{d-1'}$ and $F^{k-1'}$ meets every $[(k+2)/2]$-face of $F^{k'}$ for all $3 \leqq k \leqq d-1$.*

PROOF. The first assertion is obvious. For the second assertion, we first observe that there are two types of $[(k+2)/2]$-faces of $F^{k'}$.

(i) Faces that are the result of a $[(k+2)/2]$-face of F^k being split.

(ii) Faces of the form $G \cap \pi$ where G is a $([(k+2)/2]+1)$-face of F^k. For $F^{k-1'}$ to miss a face X of type (i) we would have to have that $X \cap F^{k-1}$ lies entirely on one side of π. Since v lies in F^{k-1}, we see that this intersection must in fact be a face of F^{k-2}. But since π cuts every $([(k+2)/2]-1)$-face of F^{k-2}, we see that the intersection cannot lie entirely on one side of π.

Suppose that $F^{k-1'}$ misses a face Z of type (ii). Since G is $([(k+2)/2]+1)$-dimensional, it intersects F^{k-1} on a $[(k+2)/2]$-face H. The face H intersects F^{k-2} on a $([(k+2)/2]-1)$-face (by induction F^{k-2} meets every $[(k+1)/2]$-face of F^{k-1} and therefore meets every $[(k+2)/2]$-face of F^{k-1}). We know that π cuts every $([(k+2)/2]-1)$-face of F^{k-2}, thus when G is cut it will have vertices both above and below π. In particular, vertices in $F^{k-2'}$, thus Z meets $F^{k-1'}$.

It is now very simple to construct our dual neighborly d-polytopes. We begin with the simplex and choose a nested set of faces $S^2 \subset S^3 \subset \cdots \subset S^{d-1}$ and apply a splitting as described above. We use the set of faces $S^{2'}, \cdots, S^{d-1'}$ to determine the next splitting, and so on.

THEOREM 3. *The polytopes generated in this way are *-neighborly.*

PROOF. This is clearly true for the simplex. Proceeding by induction, suppose we apply the splitting to a polytope P to get a polytope P'. We must show that given any $[d/2]$ facets of P', they have nonempty intersection. By induction, P has this property. If the $[d/2]$ facets chosen do not contain among them either of the two new facets created by the splitting, then they will still have nonempty intersection.

Suppose we choose $[d/2]$ facets, one of which is a new facet K, resulting from the splitting of a facet F. In P these facets, other than K, intersect on a $d-[d/2]+1$-face, that is, a $[(d+2)/2]$-face of F. Since each such face of F is split, the intersection of these $[d/2]-1$ faces will meet K.

Suppose we choose $[d/2]$ facets which include among them both new facets K and J. The $[d/2]-2$ facets (excluding K and J) meet on a $d-[d/2]+2$-face, that is, a $([(d+2)/2]+1)$-face, which intersects F on a face of dimension at least $[(d+2)/2]$. Since each such face is cut when F is split, the intersection of K and J meets the intersection of the $[d/2]-2$ facets.

LEMMA 9. *In a d-s-polytope P, all the k-faces $F^3, \cdots, F^k, \cdots, F^{d-1}$ are k-s-polytopes.*

PROOF. The splitting of F^{d-1} by π induces a splitting of F^{d-2} by $\pi \cap F^{d-2}$. The description of these splittings is the same as the description of the generation of the $(d-1)$-s-polytopes except that there is a reversal of "above" and "below", and the plane is not necessarily parallel to F^{d-2} before the sequence of rotations. Neither of these differences has any effect on the combinatorial type of polytope that results.

We now turn to the question of how many combinatorial types we have constructed.

LEMMA 10. *The number of* 3-*faces of a simple* d-*polytope with* n *facets is at most* $\binom{n}{d-3}$.

PROOF. By Lemma 1, the nonempty intersection of any $d-3$ facets has dimension 3. Since each 3-face is the intersection of $d-3$ facets, we obtain the maximum number of them if each $d-3$ facets intersect on a 3-face. In this case we get $\binom{n}{d-3}$ of them.

THEOREM 4. *The number of combinatorial types of* d-s-*polytopes is at least*

$$T(n)=\frac{(2n-4)!}{n!(n-2)!\binom{n}{d-3}}.$$

PROOF. The proof is the same as in Theorem 1 except that the number of ways that we could choose a 3-face of a d-polytope with n facets, to be a root is at most $\binom{n}{d-3}$ while the number of combinatorial types for these 3-dimensional roots is at least (and in fact greater than) $(2n-4)!/n!(n-2)!$.

In Theorem 4 we used some crude approximations. There is a formula for the number of 3-faces of a dual of a neighborly d-polytope (see [3], Ch. 9), but it is complicated and adds nothing to the result, since we have no reason to believe that these methods yield a formula that is close to the actual number of combinatorial types. The result is good enough that we know that the number of types grows large without bound as n increases. We also have

COROLLARY 1. *The number of neighborly* d-*polytopes with* n *vertices is at least* $T(n)$.

5. Diameters of d-s-polytopes

One of the important problems about simple d-polytopes that has not been settled is the Hirsch Conjecture, that the diameter of a simple d-polytope with n facets is at most $n-d$. The *diameter* is defined to be the maximum distance between vertices, where the *distance* between two vertices is the minimum number of edges of any path joining them. The Hirsch conjecture has been proved only for polytopes of dimension at most three [6].

Since the duals of the neighborly polytopes maximize the number of vertices, given the number of facets [8], one might guess that if there exist counter-examples to the Hirsch conjecture then some would be duals of neighborly polytopes. Klee has shown that the duals of the cyclic d-polytopes satisfy the Hirsch conjecture [5]. We shall show here that all d-s-polytopes also satisfy the Hirsch conjecture, thus lending a little more support to a conjecture whose truth has seemed more and more doubtful recently (see [7] and [10]).

THEOREM 5. *Let P be a d-s-polytope with n facets and let x and y be two vertices of P. If neither x nor y is on F^{d-1}, the vertices x and y can be joined by a path of length at most $n-d$ that avoids F^{d-1}. If x is not on F^{d-1} but y is, then there is a path of length at most $n-d$ that joins x and y, and meets F^{d-1} only at y.*

PROOF. Our proof is by induction on n. Since in the d-simplex, each two vertices are joined by an edge, the theorem is true for the minimum value of n. Suppose we split a facet of a d-s-polytope P, producing the d-s-polytope P' with n facets, and suppose that the theorem is true for P. Let x and y be two vertices of P', neither of which is on $F^{d-1'}$. In this case x and y are vertices of P. By induction x and y can be joined by a path of length at most $n-d-1$, which either misses F^{d-1}, or meets it only at y. In either case, the hyperplane π misses every edge of the path, thus the same path joins x and y in P'.

Now, let x and y be two vertices of P', with x not on $F^{d-1'}$, but with y on $F^{d-1'}$. There are two cases:

Case I. y is a vertex of F^{d-1}. In this case the two vertices can be joined by a path of length at most $n-d-1$ in P, and by the above argument, the same path will join them in P'.

Case II. y is a vertex of the subfacet that belongs to the two new facets in P'. In this case the vertex y is the intersection of π with an edge uv with u in $F^{d-1'}$, and v not in $F^{d-1'}$. In P, there is a path of length at most $n-d-1$ from x to v, that meets F^{d-1} only at v. We take this path in P' and add the edge vy. This produces a path of length at most $n-d$ joining x and y, and meeting $F^{d-1'}$ only at y.

COROLLARY 2. *The diameter of a d-s-polytope with n facets is at most $n-d$.*

PROOF. The proof is by induction on d. For $d=4$, Theorem 5 gives the desired path unless both vertices are on F^3. If x and y are on F^3 then since F^3 is a simple 3-polytope with $n-1$ facets, there is a path joining x and y in F^3 of length at most $n-4$ (the Hirsch conjecture is true for 3-polytopes).

Proceeding by induction, if P is a d-s-polytope, Theorem 5 establishes the truth of this theorem unless x and y are both on F^{d-1}. The same proof as above, using the inductive assumption that F^{d-1} satisfies the Hirsch conjecture, finishes the proof.

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